Indian Statistical Institute, Bangalore

M. Math First Year

Second Semester - Differential Geometry I

Midterm Exam — Date: March 01, 2018

Maximum marks: 50 — Duration: 3 hours

Attempt all problems

1. A unit speed plane curve r has the property that its tangent vector $t(s)$ makes a fixed angle θ with $r(s)$ for all s. Show that
   (a) if $\theta = 0$, then r is part of a straight line.
   (b) if $\theta = \frac{\pi}{2}$, then r is a circle.
   (c) if $0 < \theta < \frac{\pi}{2}$, then r is a logarithmic spiral. [$\tilde{r}(t) = (e^{kt}\cos t, e^{kt}\sin t)$ is a logarithmic spiral]. [10]

2. (a) Let r be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then show that the image of r is contained in a plane if and only if torsion τ is zero at every point of the curve. [8]
   (b) Show that the curve $r(t) = (\frac{1+t^4}{t^3}, t+1, \frac{t^4-7}{t^3})$ is planar. [2]

3. (a) Define vertex of curve. Show that the lim a con $r(t) = ((1 + 2\cos t)\sin t, (1 + 2\cos t)\cos t)$ has only two vertices. [6]
   (b) State the four vertex theorem. Explain why does the lim a con not contradict the four vertex theorem. [4]

4. Let $r : (a, b) \to \mathbb{R}^2$ is a regular curve and let $t_0 \in (a, b)$ such that $|r(t_o)| = \max(|r(t)| : t \in (a, b))$. Prove that $|k(t_o)| \geq \frac{1}{|r(t_o)|}$. [8]

5. Let $r(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Then show that its torsion τ is given by

$$\tau = \frac{(\dot{r} \times \ddot{r}) \cdot \dddot{r}}{\|\dot{r} \times \ddot{r}\|^2}.$$

[8]

6. (a) Compute the unit tangent, signed curvature, signed normal and torsion for the curve. $r(t) = (\frac{4}{5}\cos t, 1 - \sin t, \frac{-3}{5}\cos t)$. [4]
   (b) Verify the Frenet-Serret equations for the above curve. [4]